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Weighted domination of cocomparability graphs

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Abstract

It is shown in this paper that the weighted domination problem and its three variants, the weighted connected domination, total domination, and dominating clique problems are NP-complete on cobipartite graphs when arbitrary integer vertex weights are allowed and all of them can be solved in polynomial time on cocomparability graphs if vertex weights are integers and less than or equal to a constant c. The results are interesting because cocomparability graphs properly contain cobipartite graphs and the cardinality cases of the above problems are trivial on cobipartite graphs. On the other hand, an $O(|V|^2)$ algorithm is given for the weighted independent perfect domination problem of a cocomparability graph $G = (V, E)$.

1. Introduction

A *comparability graph* is a graph $G = (V, E)$ whose vertex set has a *transitive ordering*, i.e., a numbering $1, 2, \ldots, n$ of V such that $i < j < k, (i, j) \in E$, and $(j, k) \in E$ imply $(i, k) \in E$. There is an $O(|V|^{2.37})$ algorithm [19] to test if a graph is a comparability graph. In the case of a positive answer, an algorithm produces a transitive ordering in $O(|V| + |E| \log |V|)$ time [16]. A *cocomparability graph* is the complement of a comparability graph, or, equivalently, a graph whose vertex set has a *cocomparability ordering*, which is a numbering $1, 2, \ldots, n$ of V such that $i < j < k$ and $(i, k) \in E$ imply $(i, j) \in E$ or $(j, k) \in E$. Throughout the paper, we assume that vertices of a cocomparability graph are numbered in cocomparability ordering with $1, 2, \ldots, n$. A *dominating set* of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex in D. The *domination problem* is to find a minimum dominating set of the given graph. Suppose that every vertex $v \in V$ is associated with a weight, denoted by $w(v)$. The *weighted domination problem* involves finding a

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dominating set D of the given graph such that its weight $w(D) = \Sigma\{w(v): v \in D\}$ is minimum. The domination problem is just the weighted domination problem with $w(v) = 1$ for each vertex v. A dominating set D is *independent*, *connected*, or *total* if the subgraph induced by D has no edge, is connected, or has no isolated vertex, respectively. A set D of vertices is called a dominating clique if D is a dominating set and the subgraph induced by D is complete. Lot of work has been done to clarify the algorithmic complexity of these problems when restricted to special classes of graphs. For an overview see [6].

Many NP-complete graph problems become tractable when restricted to special subclasses of perfect graphs. This motivates the search for larger classes for which the problem is still tractable. Interval and permutation graphs are two well-known graph classes that admit many polynomial-time algorithms for NP-complete graph problems. However, if we generalize both in a natural way to chordal and comparability graphs respectively, many problems become NP-complete. Recently, attention has been drawn to cocomparability graphs, a class of graphs that properly contains interval and permutation graphs. It seems very likely that problems solvable in polynomial time on interval and permutation graphs remain tractable on cocomparability graphs. For example, hamiltonian cycle and hamiltonian path problems were shown solvable in polynomial time [7–9]. The domination problem and its variants, independent, connected and total domination problems are also solvable in polynomial time, except that finding a minimum cardinality dominating clique is NP-hard [14]. In addition, the weighted independent perfect domination problem is solvable in polynomial time [4].

We note that the domination, connected and total domination problems are NP-complete on chordal graphs. The independent domination problem is solvable in linear time on chordal graphs [10]. But it becomes NP-complete when arbitrary integer vertex weights are allowed [3]. On the other hand, the weighted independent domination problem is solvable in polynomial time on cocomparability graphs even when arbitrary vertex weights are allowed [14]. Whether the weighted domination, connected and total domination problems are solvable in polynomial time on cocomparability graphs was left unresolved in [14]. One is tempted to conjecture that they are solvable in polynomial time. Our studies showed the conjecture was not correct. In Section 2, we prove that the weighted domination, connected domination, total domination, and dominating clique problems are NP-complete on cobipartite graphs when arbitrary integer vertex weights are allowed. Since cobipartite graphs are properly contained in cocomparability graphs, all these problems are NP-complete on cocomparability graphs. The result is interesting not because it is difficult to prove but because the cardinality cases of all the above four problems are trivial on cobipartite graphs. In Section 3, we give a polynomial-time algorithm for the weighted domination problem on cocomparability graphs with bounded integer vertex weights i.e., the weight of every vertex is an integer and is less than or equal to a constant c. In Sections 4 and 5, we show how to modify the algorithm to solve the weighted total domination and connected domination problems on cocomparability graphs with bounded integer vertex weights, respectively. We do not claim that our algorithms are practical; however we feel they are of theoretical interest in demarcating the P–NP borderline of the considered problems.

The running times of the algorithms given by Chang et al. [4] for the weighted independent perfect domination problem are $O(|V||E|)$ and $O(|V|^{2.37})$, respectively. The algorithms suffer from determining whether two vertices have a neighbor in common for every pair of non-adjacent vertices. In Section 6, we propose an $O(|V|^2)$ algorithm that avoids this bottleneck of their algorithms.

2. The NP-completeness results

A graph is *cobipartite* if it is the complement of bipartite graph. In this section, we prove that all of the weighted domination, connected domination, total domination, and dominating clique problems are NP-complete for cobipartite graphs when arbitrary integer vertex weights are allowed. Note that the cardinality cases of these problems can be solved easily on cobipartite graphs. These problems can be formulated as follows:

Weighted domination (connected domination, total domination, dominating clique) problem

Instance: Graph $G = (V, E)$, each vertex $v \in V$ is associated with a weight $w(v)$; a number W, $W \leqslant \Sigma\{w(v): v \in V\}$.

Question: Is there a subset $D \subseteq V$ with $w(D) \leqslant W$ such that D is a dominating set (a connected dominating set, a total dominating set, or a dominating clique) of G?

The reduction of the proof is from the hitting set problem (See [SP8] of [12]).

Hitting set problem

Instance: Collection C of subsets of a finite set S, positive integer $K \leqslant |S|$.

Question: Is there a subset $S' \subseteq S$ with $|S'| \leqslant K$ such that S' contains at least one element of each subset in C?

Theorem 1. *The weighted domination, connected domination, total domination, and dominating clique problems on cobipartite graphs are* NP-*complete.*

Proof. Clearly, the weighted domination, connected domination, total domination, and dominating clique problems are in NP. We transform the hitting set problem to these problems. For an instance of the hitting set problem we construct graph $G = (V, E_{SS} \cup E_{SC} \cup E_{CC})$ where $V = S \cup C$, $E_{SS} = \{(s_i, s_j): s_i, s_j \in S,\ i \neq j\}$, $E_{SC} = \{(s_i, C_j): s_i \in S,\ C_j \in C,\ s_i \in C_j\}$, and $E_{CC} = \{(C_i, C_j): C_i,\ C_j \in C,\ i \neq j\}$. The weight of each vertex in V is as follows: $w(s_i) = 1$ for all $s_i \in S$ and $w(C_i) = |S| + 1$ for all $C_i \in C$. It is straightforward to verify that G is a cobipartite graph since both S and

C are complete subgraphs in G, and G can be considered as the complement of a bipartite graph with two independent vertex sets S and C.

If the hitting set problem instance has a hitting set D of size at most K, then D is a dominating set of G too. Obviously, the induced subgraph of D is complete and thus has no isolated vertices. In other words, we have a dominating set, a connected dominating set, a total dominating set, a dominating clique of weight K. On the other hand, suppose G has a dominating set D of weight K where $K \leqslant |S|$. Since the weight of any vertex in C is greater than $|S|$, $D \subseteq S$ and hence the induced subgraph of D is complete and thus has no isolated vertices. Since D dominates C, each $C_i \in C$ has at least one element in D. Hence D is a hitting set of size K of the hitting set problem instance. □

3. Weighted domination

In this section, we give a polynomial-time algorithm for the weighted domination problem on cocomparability graphs where vertex weights are bounded integers. Manacher and Mankus showed that any algorithm for finding a minimum weighted dominating set, total dominating set, and connected dominating set for nonnegative weights can be extended to handle negative weights without loss of efficiency [17]. Thus, we assume that $G = (V, E)$ is a cocomparability graph, and for every $v \in V$, $w(v)$ is an integer and $0 \leqslant w(v) \leqslant c$ where c is a constant. For technical reasons, we add two isolated vertices, 0 and $n + 1$ with $w(0) = w(n + 1) = 0$ to G to obtain a new cocomparability graph G_a with a cocomparability ordering $0, 1, 2, \ldots, n, n + 1$. Note that D is a dominating set of G if and only if $D \cup \{0, n + 1\}$ is a dominating set of G_a. For simplicity, we assume that G is the new graph added with 0 and $n + 1$. We need some notation.

- For a vertex i, let $N(i)$ denote the set of vertices adjacent to i, and $N[i] = N(i) \cup \{i\}$. Vertex j is a *right neighbor* (resp. *left neighbor*) of vertex i if vertex j is adjacent to vertex i and $j > i$ (resp. $j < i$). Let $N^+[i]$ denote the set containing vertex i and all right neighbors of i, and $N^-[i]$ is the set containing vertex i and all left neighbors of i.
- For two vertices i and j where $i < j$, $B(i, j)$ is the set of all vertices between i and j including i and j. That is, $B(i, j) = \{k : i \leqslant k \leqslant j\}$. $B(i, j)$ is called a *consecutive vertex set*. Let $N[i, j]$ denote the set of all vertices k of $N[i] \cup N[j]$ such that $i \leqslant k \leqslant j$.
- For a subset X of V, let $N[X] = \bigcup_{i \in X} N[i]$, $ZW(X)$ denote the set of zero weight vertices in X, $\min(X)$ and $\max(X)$ denote the leftmost and rightmost vertices of X in cocomparability ordering, respectively, and let $G[X]$ denote the induced subgraph of X in graph G.
- For a vertex i (resp. a set X of vertices) and a set D of vertices, a vertex j in $N[i] - N[D - \{i\}]$ (resp. $N[X] - N[D - X]$) is called a *private neighbor of i (resp. X) with respect to D*. We may say that vertex j is a private neighbor of i (resp. X) if

D is understood without ambiguity. If $i \in D$ and i has no private neighbor with respect to D, then $D - \{i\}$ still dominates the set of vertices dominated by D. In this case, we say that i is redundant in D.

The following lemma can be proved easily.

Lemma 1. *Suppose G is a cocomparability graph, S is a subset of vertices of G and $G[S]$ is connected. Then, S dominates the vertex set $\{\min(S), \min(S)+1, \ldots, \max(S)\}$.*

Let D be a minimum weighted dominating set of a cocomparability graph G. Suppose C_1 and C_2 are any two connected components of $G[D]$ and $\max(C_1) < \max(C_2)$. Following the above lemma, it is easy to see that $\max(C_1) < \min(C_2)$. Thus, the connected components of $G[D]$ can be sorted in an ordering $C_1, C_2, \ldots, C_q$, where $C_1 \equiv \{0\}$, $C_q \equiv \{n+1\}$, and q is the number of all connected components of $G[D]$, such that $\min(C_1) \leqslant \max(C_1) < \min(C_2) \leqslant \max(C_2) < \cdots \leqslant \max(C_{q-1}) < \min(C_q) \leqslant \max(C_q)$.

For each connected component C of $G[D]$, let $S(C)$ be the set of vertices on a shortest path from $\min(C)$ to $\max(C)$ in the induced subgraph $G[C]$. We call set $N^+[\min(C)] \cap C$ (resp. $N^-[\max(C)] \cap C$) the *left part* (resp. *right part*) of C. For clarity, we denote the left part (resp. right part) of C by $XL(C)$ (resp. $XR(C)$). Note that $D \cup ZW(V)$ is also a minimum weighted dominating set. Thus, there exists a minimum weighted dominating set D such that $ZW(V) \subseteq D$.

Lemma 2. *Suppose D is a minimum weighted dominating set of cocomparability graph G where $ZW(V) \subseteq D$ and $G[D]$ has q connected components $C_1, C_2, \ldots, C_{q-1}, C_q$, where $C_1 \equiv \{0\}$ and $C_q \equiv \{n+1\}$, sorted in increasing order of the rightmost vertices of connected components. For $1 < r \leqslant q$, we have:*

(1) *$XR(C_{r-1}) \cup XL(C_r)$ dominates $B(\max(C_{r-1}), \min(C_r))$.*

(2) *$(D - C_r) \cup S(C_r) \cup XL(C_r) \cup XR(C_r)$ is a dominating set of G.*

(3) *If $|S(C_r)| \leqslant 3$, then $w(C_r - S(C_r)) \leqslant 4c$.*

(4) *If $|S(C_r)| > 3$, then $XL(C_r) \cap XR(C_r) = \emptyset$.*

(5) *Suppose $|S(C_r)| > 3$ and $S(C_r) = i \to i' \to h \to \cdots \to k \to j' \to j$. Then $w(XL(C_r) - \{i, i'\}) \leqslant 2c$ and $w(XR(C_r) - \{j, j'\}) \leqslant 2c$.*

Proof. In the following, we prove the above statements one by one.

(1) For simplicity, let $i = \max(C_{r-1})$ and $j = \min(C_r)$. Clearly, $i < j$. Suppose $k \in B(i,j)$, $i < k < j$, and k is adjacent to neither i nor j. Since D is a dominating set of G, k is adjacent to a vertex $h \in D$. Obviously, $h < i$ or $h > j$. By cocomparability ordering, h is adjacent to i if $h < i$ and h is adjacent to j if $h > j$. By definition, $h \in XR(C_{r-1})$ if h is adjacent to i, and $h \in XL(C_r)$ if h is adjacent to j. Thus, $XR(C_{r-1}) \cup XL(C_r)$ dominates $B(i,j)$.

(2) For simplicity, let $i = \min(C_r)$ and $j = \max(C_r)$. Suppose $k \in N[C_r]$. There are three cases. Case1, $i \leqslant k \leqslant j$. By Lemma 1, $k \in N[S(C_r)]$ in this case. Case 2, $k < i$. Let h be any vertex that is in C_r and is adjacent to k. Obviously, $i \leqslant h$. By cocomparability

ordering, either k is adjacent to i or h is adjacent to i. By definition, $k \in N[XL(C_r)]$. Case 3, $k > j$. In this case, we can prove that $k \in N[XR(C_r)]$ by arguments similar to those for Case 2. We have proved that if $k \in N[C_r]$, then $k \in N[S(C_r) \cup XL(C_r) \cup XR(C_r)]$. Clearly, $(D - C_r) \cup S(C_r) \cup XL(C_r) \cup XR(C_r)$ is also a minimum weighted dominating set of G.

(3) Suppose $w(C_r - S(C_r)) > 4c$. For simplicity, let $i = \min(N[C_r])$, $j = \max(N[C_r])$. By definition, $N[C_r] \subseteq B(i, j)$. Let h and k be two vertices of C_r that are adjacent to i and j, respectively. Clearly, h, $k \in N[S(C_r)]$. Hence $G[S(C_r) \cup \{i, j, h, k\}]$ is connected. By Lemma 1, $S(C_r) \cup \{i, j, h, k\}$ dominates $B(i, j)$. Hence, $(D - C_r) \cup S(C_r) \cup \{i, j, h, k\}$ is a dominating set of G. Obviously, $w(\{i, j, h, k\}) \leqslant 4c$. That is, the weight of $(D - C_r) \cup S(C_r) \cup \{i, j, h, k\}$ is less than $w(D)$, a contradiction of D as a minimum weighted dominating set.

(4) Proved by the definition of $S(C_r)$.

(5) For simplicity, let $T = XL(C_r) - \{i, i'\}$. Suppose $w(T) > 2c$. Let x be any vertex in $N[T] - N[D - T]$. It is not hard to verify that $x < i$. Let $u = \min(N[T] - N[D - T])$. Then, $N[T] - N[D - T] \subseteq B(u, i)$. Let v be any vertex in T that is adjacent to u. Clearly, $v > i$. By Lemma 1, $\{u, v\}$ dominates $B(u, i) \subset B(u, v)$. Hence $(D - T) \cup \{u, v\}$ is a dominating set. This contradicts the fact that D is a minimum weighted dominating set since $w(\{u, v\}) \leqslant 2c < w(T)$. This proves that $w(T) \leqslant 2c$. By symmetry, we can prove that $w(XR(C_r) - \{j, j'\}) \leqslant 2c$. □

The next lemma immediately follows from Lemma 2.

Lemma 3. *Given a cocomparability graph G, there exists a minimum weighted dominating set D such that, for every connected component C of $G[D]$,*

(1) *if $|S(C)| \leqslant 3$, then*

(i) *$w(C) \leqslant 7c$ and*

(ii) *$ZW(N^+[\min(C)]) \cup ZW(N^-[\max(C)]) \subseteq C$; and*

(2) *if $|S(C)| > 3$, then C can be partitioned into three disjoint subsets $S(C) - \{i, i', j, j'\}$, the left part, and the right part of C, where $S(C_r) = i \to i' \to h \to \cdots \to k \to j' \to j$, such that the left part satisfies the following two conditions,*

(i) *$ZW(N^+[\min(C)]) \subseteq XL(C)$ and*

(ii) *$w(XL(C)) \leqslant 4c$;*

and the right part satisfies the following two conditions,

(i) *$ZW(N^-[\max(C)]) \subseteq XR(C)$ and*

(ii) *$w(XR(C)) \leqslant 4c$.*

Our algorithm solves the problem by constructing a weighted directed graph $G'' = (V'', E'')$ such that a minimum weighted dominating set D of cocomparability graph $G = (V, E)$, satisfying the conditions given in Lemma 3, corresponds to a minimum weighted path of G''. Note that we refer to an element of V'' as a node and an element of V as a vertex for clarity. Each node of G'' corresponds to a set of vertices of G. The basic ideas for constructing G'' are as follows.

- For each $0 < i < n+1$, we construct a node, v_i.
- For each subset X of $N^-[i]$ (resp. $N^+[i]$) where $0 < i < n+1$, $i \in X$, $w(X) \leqslant 4c$, and $ZW(N^-[i]) \subseteq X$ (resp. $ZW(N^+[i]) \subseteq X$), we construct a node X. Let $XR(i)$ (resp. $XL(i)$) be the set of all such nodes.
- For each i, $0 \leqslant i \leqslant n+1$, construct a node $Z1(i)$.
- For each subset Z of $N[i,j]$ of two vertices i and j where $0 < i < j < n+1$, $w(Z) \leqslant 7c$, $i, j \in Z$, $ZW(N^+[i]), ZW(N^-[j]) \subseteq Z$, and $N^+[i] \cap N^-[j] \cap Z \neq \emptyset$, we construct a node Z. Clearly, $G[Z]$ is connected and $1 < |S(Z)| \leqslant 3$. Let $Z2(i,j)$ be the set of all such nodes.
- Let $V'' = V' \cup XR \cup XL \cup Z1 \cup Z2$ where
  $V' = \{v_i : 0 < i < n+1\}$,
  $XR = \bigcup_{1 \leqslant i \leqslant n} XR(i)$,
  $XL = \bigcup_{1 \leqslant i \leqslant n} XL(i)$,
  $Z1 = \{Z1(i) : 0 \leqslant i \leqslant n+1\}$, and
  $Z2 = \bigcup_{1 \leqslant i < j \leqslant n} Z2(i,j)$.
  Clearly, $|V''| = O(|V|^{7c})$.

Next, we show how to construct edges between two nodes of V''. Let the edge of G'' directed from node X to node Y be denoted by $\langle X, Y \rangle$. We will explain the reasons for constructing these directed edges later.

- For each edge (i,j) in G, we construct two directed edges $\langle v_i, v_j \rangle$ and $\langle v_j, v_i \rangle$. Let $E_d = \{\langle v_i, v_j \rangle, \langle v_j, v_i \rangle : v_i, v_j \in V', \text{ and } (i,j) \in E\}$.
- For each $X \in XL(i)$ and $v_j \in V'$ where $0 < i < j \leqslant n$, $j \notin N(i)$, and $N(j) \cap X \neq \emptyset$, we construct a directed edge $\langle X, v_j \rangle$. Let E_L be the set of all such directed edges in G''. By symmetry, we construct the set E_R of directed edges in G'' where $E_R = \{\langle v_i, X \rangle : v_i \in V', X \in XR(j), 0 < i < j \leqslant n, i \notin N(j), N(i) \cap X \neq \emptyset\}$.
- For $X_1 \in XL(i)$ and $X_2 \in XR(j)$ where $0 < i < j < n+1$, $(i,j) \notin E$, $X_1 \cap X_2 = \emptyset$, $\max(X_1) < j$, $i < \min(X_2)$, and there exist a vertex $h \in X_1$ and a vertex $k \in X_2$ such that $(h,k) \in E$, we construct a directed edge $\langle X_1, X_2 \rangle$. Let E_{LR} be the set of all such directed edges in G''.
- For $X_1 \in XR(i)$ and $X_2 \in XL(j)$ where $0 < i < j < n+1$, and $X_1 \cup X_2$ dominates $B(i,j)$, we construct a directed edge $\langle X_1, X_2 \rangle$. Let E_{RL} be the set of all such directed edges in G''.
- Other sets of directed edges of G'' are E_{RZ}, R_{ZL}, and E_Z. An edge of E_{RZ} is directed from a node of XR to a node of $Z1 \cup Z2$. An edge of E_{ZL} is directed from a node of $Z1 \cup Z2$ to a node of XL. An edge of E_Z is directed from a node of $Z1 \cup Z2$ to another node of $Z1 \cup Z2$. They are formally defined in the following:
  $E_{RZ} = \{\langle X, Z \rangle : X \in XR, \quad Z \in Z1 \cup Z_2, \quad \max(X) < \min(Z), \quad X \cup Z \text{ dominates } B(\max(X), \min(Z))\}$,
  $E_{ZL} = \{\langle Z, X \rangle : X \in XL, \quad Z \in Z1 \cup Z_2, \quad \max(Z) < \min(X), \quad X \cup Z \text{ dominates } B(\max(Z), \min(X))\}$,
  and
  $E_Z = \{\langle Z_1, Z_2 \rangle : Z_1, Z_2 \in Z1 \cup Z_2, \quad \max(Z_1) < \min(Z_2), \quad Z_1 \cup Z_2 \text{ dominates } B(\max(Z_1), \min(Z_2))\}$.

- Finally, we let $E'' = E_d \cup E_L \cup E_R \cup E_{LR} \cup E_{RL} \cup E_{RZ} \cup E_{ZL} \cup E_Z$.

Each node $q \in V''$ corresponds to a set of vertices in V, denoted by $V(q)$. Define the weight $w(q)$ of each node $q \in V''$ to be the total weight of vertices in $V(q)$, i.e. $w(q) = w(V(q))$. For a set Q of nodes of V'', define $V(Q) = \bigcup_{q \in Q} V(q)$ and $w(Q) = \sum_{q \in Q} w(q)$. For simplicity, we use q to denote $V(q)$ for a node q of G'' if there is no ambiguity.

Let D be a minimum weighted dominating set satisfying the conditions given in Lemma 3 and C be a connected component of $G[D]$, $i = \min(C)$, and $j = \max(C)$. We have the following observations.

- If $C = \{i\}$, then C corresponds to node $Z1(i)$. If $1 < |S(C)| \leqslant 3$, then C corresponds to a node of $Z2(i, j)$ since $w(C) \leqslant 7$.
- If $|S(C)| = 4$, then C can be partitioned into two disjoint parts, the left and the right parts, where the left part corresponds to a node X of $XL(i)$ and the right part corresponds to a node Y of $XR(j)$, and there exists an edge $\langle X, Y \rangle \in E_{LR}$.
- If $|S(C)| > 4$, then C can be partitioned into three disjoint parts such that one part (the left part) corresponds to a node X of $XL(i)$, the second part (right part) corresponds to a node Y of $XR(j)$, the third part is $S(C) - (X \cup Y)$ which corresponds to a path from node v_h to node v_k where $S(C) = i \to i' \to h \to \cdots \to k \to j' \to j$, and there exist edges $\langle X, v_h \rangle \in E_L$, $\langle v_k, Y \rangle \in E_R$. In other words, if $|S(C)| > 4$, then C corresponds to a path from node X to node Y.

Suppose $G[D]$ has q connected components, $C_1 \equiv \{0\}$, $C_2, \ldots, C_q \equiv \{n+1\}$, sorted in increasing order of $\max(C_r)$'s for $1 \leqslant r \leqslant q$. Clearly C_1 and C_q correspond to nodes $Z1(0)$ and $Z1(n+1)$, respectively. Consider any two consecutive connected components C_{r-1} and C_r where $1 < r \leqslant q$. We have the following observations.

- If both $|S(C_{r-1})| \leqslant 3$ and $|S(C_r)| \leqslant 3$, then there is a directed edge from the node corresponding to C_{r-1} to the node corresponding to C_r (See the definition of E_Z).
- If $|S(C_{r-1})| \leqslant 3$ and $|S(C_r)| > 3$, then there is a directed edge from the node corresponding to C_{r-1} to the node corresponding to the left part of C_r (See the definition of E_{ZL}).
- If $|S(C_{r-1})| > 3$ and $|S(C_r)| \leqslant 3$, then there is a directed edge from the node corresponding to the right part of C_{r-1} to the node corresponding to C_r (See the definition of E_{RZ}).
- If both $|S(C_{r-1})| > 3$ and $|S(C_r)| > 3$, then there is a directed edge from the node corresponding to the right part of C_{r-1} to the node corresponding to the left part of C_r (See the definition of E_{RL}).

Following the above observations, we have the next lemma.

Lemma 4. *For a minimum weighted dominating set D of G, satisfying the conditions given in Lemma 3, there exists a path P from $Z1(0)$ to $Z1(n+1)$ in G'' where $w(P) = w(D)$.*

Lemma 5. *If P is a directed path from $Z1(0)$ to $Z1(n+1)$ in G'', then $V(P)$ is a dominating set of G.*

Proof. We note that there are no edges (1) directed from a node of XL to a node of $Z1 \cup Z2$, (2) directed from a node of V' to a node of $Z1 \cup Z2 \cup XL$, and (3) directed from a node of $Z1 \cup Z2$ to a node of $XR \cup V'$. Thus, P can be considered as the following path,

$$P_1 \to P_2 \to \cdots \to P_p,$$

where each subpath P_r, $1 \leqslant r \leqslant p$, is a maximal sub-path of P such that (1) all nodes in P_r are in $Z1 \cup Z2$ or (2) P_r starts from a node of XL to a node of XR and all their internal nodes are in V'. Obviously, the path start of P_1 is node $Z(0)$, and the path end of P_p is node $Z(n+1)$.

Suppose P_r is a maximal sub-path that starts from a node of XL to a node of XR and all their internal nodes are in V'. Let the path start, $X_{r,s}$, of P_r be a node in $XL(f)$ of some vertex f of V and the path end, $X_{r,t}$, of P_r be a node in $XR(k)$ of some vertex k of V. By definition, it is easy to see that there exists a path from f to k in $G[V(P_r)]$. By Lemma 1, $V(P_r)$ dominates $\{v : v \in V, f \leqslant v \leqslant k\}$ in G. On the other hand, suppose P_r is a maximal sub-path that consists of nodes in $Z1 \cup Z2$ and it starts from node $X_{r,s}$ to node $X_{r,t}$. Suppose $X_{r,s} = Z1(f)$ or $X_{r,s} \in Z2(f, g)$ and $X_{r,t} = Z1(k)$ or $X_{r,t} \in Z2(h, k)$. By definition, it is straightforward to verify that $V(P_r)$ dominates $\{u : u \in V, f \leqslant u \leqslant k\}$.

Consider two sub-paths P_{r-1} and P_r for $1 < r \leqslant p$. There are three cases.

Case 1: The path end, $X_{r-1,t}$, of P_{r-1} is node $Z1(k)$ or a node of $Z2(h, k)$. In this case, it is easy to see that the path start, $X_{r,s}$ of P_r is a node of $XL(f)$ of some f of V. By the definition of E_{ZL}, $k < f$ and $V(X_{r-1,t} \cup X_{r,s})$ dominates $B(k, f)$.

Case 2: The path start, $X_{r,s}$ of P_r is node $Z1(h)$ or a node of $Z2(h, k)$. In this case, it is easy to see that the path end $X_{r-1,t}$ of P_{r-1} is a node of $XR(f)$of some f of V. By the definition of E_{RZ}, $f < h$ and $V(X_{r-1,t} \cup X_{r,s})$ dominates $B(f, h)$.

Case 3: The path end, $X_{r-1,t}$, of P_{r-1} is a node of $XR(f)$ of some f of V and the path start $X_{r,s}$, of P_r is a node of $XL(k)$ of some k of V. By the definition of E_{RL}, $f < k$ and $V(X_{r-1,t} \cup X_{r,s})$ dominates $B(f, k)$.

Following the above observations, it is straighforward to verify that $V(P)$ dominates $\{u : 0 \leqslant u \leqslant n+1\}$. Thus $V(P)$ is a dominating set of G. This completes the proof. $\square$

Theorem 2. *A minimum weighted dominating set of a cocomparability graph G, where vertex weights are bounded integers, can be found in polynomial time.*

Proof. Immediately following from Lemmas 4 and 5, we can find a minimum weighted dominating set of G by finding a minimum weighted path from $Z1(0)$ to $Z1(n+1)$ in G''. Since the number of nodes of graph G'' is $O(|V|^{7c})$ and the weight of any node is non-negative, it can be implemented to run in polynomial time. $\square$

The above algorithm can be slightly modified to solve the weighted total domination and weighted connected domination problems in polynomial time when restricted to cocomparability graphs with bounded integer vertex weights.

4. Weighted total domination

In this section, we show that the weighted total domination problem is also polynomial-time solvable when restricted to cocomparability graphs with bounded integer vertex weights. We use the results of the previous section. We also assume that all vertex weights are non-negative integers and $w(v) \leqslant c$ for every $v \in V$ where c is a constant. For a total domination TD of a graph G, by definition, each connected component of $G[TD]$ has at least two vertices. Thus, $D - \{0, n+1\}$ is a total dominating set of G if and only if D is a dominating set of G_a such that each connected component of $G_a[D]$ has at least two vertices except the connected components $\{0\}$ and $\{n+1\}$. This suggests that we construct a directed graph $G^t = (V^t, E^t)$ by removing all nodes in $\{Z1(i): 1 \leqslant i \leqslant n\}$ from G''. Let $V^t = V'' - \{Z1(i): 1 \leqslant i \leqslant n\}$ and G^t be the subgraph of G'' induced by V^t. Similarly, we can prove that a minimum weighted path from $Z1(0)$ to $Z1(n+1)$ in graph G^t corresponds to a dominating set D of G_a and $D - \{0, n+1\}$ is a minimum weighted total dominating set of G. Hence, a minimum weighted total dominating set in G can be found in polynomial time. Thus we have the following theorem.

Theorem 3. *A minimum weighted total dominating set of a cocomparability graph G, where vertex weights are bounded integers, can be found in polynomial time.*

5. Weighted connected domination

In this section, we show that the weighted connected domination problem is also polynomial-time solvable when restricted to cocomparability graphs with bounded integer vertex weights. We use the results of Section 3. We also assume that all vertex weights are non-negative integers and $w(v) \leqslant c$ for every $v \in V$ where c is a constant. By Lemma 5, we know that a path from node $Z(0)$ to node $Z(n+1)$ in G'' corresponds to a dominating set D in G_a. Clearly, $D - \{0, n+1\}$ is a connected dominating set of G if and only if D is a dominating set of G_a and $G[D - \{0, n+1\}]$ is connected. Since $G[D - \{0, n+1\}]$ has only one connected component, by definition, $D - \{0, n+1\}$ corresponds to a path in G'' that is either a path of single node or a path that starts from a node $X \in XL$ to a node $Y \in XR$. This suggests that we construct a directed graph $G^c = (V^c, E^c)$ by removing edges E_{RL}, E_Z and some edges of E_{ZL} and E_{RZ} from G''. The edges removed from E_{ZL} are those edges not directed from $Z1(0)$. The edges removed from E_{RZ}

are those edges not directed to node $Z1(n+1)$. Let

$$V^c = V''$$

$$E^c = E_d \cup E_L \cup E_R \cup E_{LR} \cup E_0 \cup E_{n+1},$$

where E_0 is the set of edges directed from node $Z1(0)$ and E_{n+1} is the set of edges directed to node $Z1(n+1)$ in G''.

Similarly, we can prove that a minimum weighted path P from $Z1(0)$ to $Z1(n+1)$ in G^c corresponds to a dominating set D of G_a and $D - \{0, n+1\}$ is a minimum weighted connected dominating set of graph G. Thus, we have the following theorem.

Theorem 4. *A minimum weighted connected dominating set of a cocomparability graph G, where vertex weights are bounded integers, can be found in polynomial time.*

6. Weighted independent perfect domination

A *perfect dominating set* of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to *exactly one* vertex in D. The *perfect domination problem* involves finding a minimum perfect dominating set of the given graph. Suppose that every vertex $v \in V$ is associated with a weight $w(v)$ and every edge $e \in E$ has a weight $w(e)$. The *weighted perfect domination problem* involves finding a perfect dominating set D such that its weight

$$w_e(D) = \sum \{w(v) : v \in D\} + \sum \{w(u, v) : u \notin D,\ v \in D, \text{ and } (u, v) \in E\}$$

is minimum. The perfect domination problem is just the weighted perfect domination problem with $w(v) = 1$ for each vertex v and $w(e) = 0$ for each edge e. Yen and Lee [20] proved that the perfect domination problem is NP-complete for bipartite graphs and chordal graphs. Yen and Lee [21] also considered the following variants of perfect domination. A perfect dominating set D is *independent*, *connected* or *total* if the subgraph induced by D has no edge, is connected, or has no isolated vertex, respectively. They gave NP-completeness results of these variants on bipartite graphs and chordal graphs, except for the connected domination problem on chordal graphs. On the other hand, Chang and Liu [5] gave a linear-time algorithm for the weighted connected perfect domination problem on chordal graphs by using clique-tree structures of chordal graphs. Note that independent perfect dominating set was called *efficient dominating set* by Bange et al. [1], *perfect* 1-*code* by Biggs [2] and Kratochvíl [13], *perfect* 1-*domination* by Livingston and Stout [15], *perfect domination* by Fellows and Hoover [11]. In this paper, we follow the notation given by Yen and Lee [21]. Not all graphs contain an independent perfect dominating set. An independent perfect dominating set of a graph G is also a minimum dominating set of G [1]. In this section, we give an $O(|V|^2)$ algorithm for the weighted independent perfect domination problem for a cocomparability graph $G = (V, E)$. This problem was first studied by Chang et al. [4]. Define $\bar{w}(v) = w(v) + \sum\{w(u, v): (u, v) \in E\}$. Then, for an independent perfect domination set D, $w_e(D) = \sum\{\bar{w}(v): v \in D\}$. Note that $\bar{w}(v)$ for all $v \in V$ can be computed in $O(|E|)$ time. Thus, for solving the weighted independent perfect domination problem, without loss of generality, we may assume that $w(e) = 0$ for all $e \in E$ [4]. Let $G = (V, E)$ be a cocomparability graph with a given cocomparability ordering. Note that D is an independent perfect dominating set of G if and only if $D \cup \{0, n+1\}$ is an independent perfect dominating set of G_a. For technical reasons, in this section we assume that G has been added two isolated vertices 0 and $n + 1$ with $w(0) = w(n+1) = 0$; and a cocomparability ordering $0, 1, 2, \ldots, n, n+1$ is given. For convenience we need the following notation, where v is a vertex. Some of them were defined in [4].

- For $0 \leqslant v \leqslant n+1$, $\text{high}(v) = \max(N[v])$, $\text{low}(v) = \min(N[v])$, $d^+(v) = |N^+[v]|$ $d^-(v) = |N^-[v]|$, $s^-(v) = \max(\{0, 1, 2, \ldots, v\} - N[v])$, $s^+(v) = \min(\{v, v+1, \ldots, n, n+1\} - N[v])$.
- For a vertex v, $\text{high}(v) > s^+(v)$, R_v is an $(n+2)$-dimensional vector defined as follows: $R_v(u) = 1$ if $s^+(v) < u \leqslant \text{high}(v)$ and $u \in N[v]$; $R_v(u) = 0$ otherwise.
- For a vertex v, $\text{low}(v) < s^-(v)$, L_v is an $(n+2)$-dimensional vector defined as follows: $L_v(u) = 1$ if $\text{low}(v) < u \leqslant s^-(v)$ and $u \notin N[v]$; $L_v(u) = 0$ otherwise.

The following lemma was proved in [4].

Lemma 6. $D = \{0 \equiv v_0 < v_1 < v_2 < \cdots < v_r < v_{r+1} \equiv n+1\}$ *is an independent perfect dominating set of a cocomparability graph G if and only if the following three conditions hold for all* $1 \leqslant i \leqslant r \leqslant 1$:
(1) $\text{high}(v_{i-1}) < v_i$,
(2) $v_{i-1} < \text{low}(v_i)$, *and*
(3) $\{x : x \in V, v_{i-1} \leqslant x \leqslant v_i\}$ *is the disjoint union of* $N^+[v_{i-1}]$ *and* $N^-[v_i]$.

Based upon the above lemma, $O(|V||E|)$ and $O(|V|^{2.37})$ time algorithms were proposed by Chang et al. [4]. The bottleneck of their algorithms is to check whether $N[u] \cap N[v] = \emptyset$ for each pair of non-adjacent vertices u and v. Lemma 6 can be written in the following form, which is more useful in designing an efficient algorithm.

Lemma 7. $D = \{0 \equiv v_0 < v_1 < v_2 < \cdots < v_r < v_{r+1} \equiv n+1\}$ *is an independent perfect dominating set of a cocomparability graph G if and only if, for all* $1 \leqslant i \leqslant r+1$, *one of the following two conditions holds.*
(1) $s^+(v_{i-1}) = \text{high}(v_{i-1}) + 1 = \text{low}(v_i)$ *and* $s^-(v_i) = \text{low}(v_i) - 1 = \text{high}(v_{i-1})$, *or*
(2) $v_{i-1} < s^+(v_{i-1}) = \text{low}(v_i) < \text{high}(v_{i-1}) = s^-(v_i) < v_i$ *and* $R_{v_{i-1}} = L_{v_i}$.

Working from Lemma 7, we design the following algorithm.

Algorithm WIPD. Find a weighted independent perfect dominating set of a cocomparability graph.
Input. A cocomparability graph $G = (V, E)$ with a cocomparability ordering $0, 1, \ldots, n, n+1$, in which each vertex v is associated with a weight $w(v)$.
Output. A minimum weighted independent perfect dominating set D of G.
Method.

1. weight(0) $\leftarrow 0$;
2. **for** $v = 1$ **to** $n + 1$ **do**
3. weight(v) $\leftarrow \infty$;
4. **for all** $u < v$ satisfying either
   (C1) $s^+(u) = \text{high}(u) + 1 = \text{low}(v)$ *and* $s^-(v) = \text{low}(v) - 1$ or
   (C2) $u < s^+(u) = \text{low}(v) < \text{high}(u) = s^-(v) < v$ and $R_u \equiv L_v$.
   do
5. **if** (weight(u) + $w(v)$ < weight(v))
6. **then** {weight(v) $\leftarrow$ weight(u) + $w(v)$; previous(v) $\leftarrow u$;}
   end do;
   end do;
7. $D \leftarrow \emptyset$;
8. $v \leftarrow$ previous($n + 1$);
9. **while** ($v \neq 0$) **do** {$D \leftarrow D \cup \{v\}$; $v \leftarrow$ previous(v);}

The bottleneck of this algorithm is determining whether (C2) holds. Whether (C2) holds can be checked in time $O(|V|)$ for a pair of vertices u and v by a straight-forward implementation. This leads to an $O(|V|^3)$ time algorithm. Note that low(v), high(v), $s^+(v)$ and $s^-(v)$ of a vertex $v \in V$ can be computed in time $O(|V|)$. Also, we note that there are at most $O(|V|)$ L vectors and R vectors. Each vector can be constructed in $O(|V|)$ time if low(v), high(v), $s^+(v)$ and $s^-(v)$ are available. These vectors can be sorted in lexicographic ordering in $O(|V|^2)$ time by using a radix sorting algorithm (see [18, p. 115]. The key idea in the sorting step is that vectors should not be moved away from their initial place. This can be done by using pointer. Only pointers are moved during sorting steps. After sorting, the vectors with the same value will be in consecutive positions in the sorted sequence. By comparing every two vectors adjacent in the sorted sequence, in $O(|V|^2)$ time we can partition vectors into lists such that vectors in the same list are all equal, and two vectors from two different lists are not equal. Then, for a vertex v we can find all vertices u such that $u < s^+(u) = \text{low}(v) < \text{high}(u) = s^-(v) < v$ *and* $R_u = L_v$ in $O(|V|)$ time. In other words, line 4–6 of Algorithm WIPD can be implemented in $O(|V|)$ time. Thus, we have the following theorem:

Theorem 5. *Given a weighted cocomparability graph G with a cocomparability ordering of the vertices, a minimum weighted independent perfect dominating set of G can be found in $O(|V|^2)$ time and space.*

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